A NEW METHOD FOR THE CALCULATION OF SELF-SUSTAINED OSCILLATIONS: THE PERTURBATION OF THE HELMHOLTZ MOTION

J. Kergomard, S. Divoir, S. Farner, C. Vergez

Laboratoire de Mécanique et d'Acoustique, CNRS
31 Chemin Joseph Aiguier, 13401 Marseille Cedex 20
France
kergomard@lma.cnrs-mrs.fr

ABSTRACT

When losses are ignored, elementary solutions for the classical models of self sustained instruments, such as reed or bowed string instruments, are pure square or ”rectangular” signals, called Helmholtz motion. When losses are introduced, round corner signals are obtained, and the calculation becomes delicate. Ab initio calculation is possible, but methods limited to the steady-state regime make it easier to study the influence of the parameters on the spectrum and the playing frequency: the harmonic balance is well known, but, because losses are small, another iterative technique is suggested. Considering e.g. reed instruments, the Fourier components of the input pressure signal can be divided into two parts: the components with high input impedance, and those with low input impedance (corresponding to the missing harmonics of the rectangular signal). A perturbation method can be obtained by starting from infinite and zero impedances, respectively. A key point is that at each step, frequency is fixed in order to calculate the perturbation, then a new value is calculated using any equation of the harmonic balance system, an excellent candidate being the reactive power defined by Boutillon. In this preliminary study, results are compared for a simplified problem to those of the harmonic balance method, and they are very interesting, especially far from the oscillation thresholds.

1. INTRODUCTION

Since Helmholtz, it is well known that the most common sounds of bowed string instruments correspond to a signal close to a pure square or ”rectangular” shape (for the signal of the string velocity).

When losses are ignored, solutions of elementary models are pure square or ”rectangular” signals, called Helmholtz motion. When losses are introduced, round corner signals are obtained, and the calculation becomes more delicate (see e.g. Woodhouse [1]). Complete computation in the time domain is possible, but taking into account the simplicity of the solutions when no losses are present, an interesting question is whether it is possible to deduce the result with losses from the result without losses by using a perturbation method. The present paper gives a preliminary answer for a similar, but simpler problem: the steady-state, periodic regime of the internal pressure signal for a cylindrical, clarinet-like instrument. It is simple because the nonlinear characteristic, assumed to be independent of time, can be written as a polynomial of the third order, at least at rather low levels of excitation, and because, if the playing frequency is much smaller than the eigenfrequency of the reed, the reed can be regarded as a simple spring without mass and damping. With some classical, complementary hypotheses (see e.g. Kergomard [2]), the system to be solved is a system of two equations relating the acoustic volume velocity $u(t)$ at the input of the resonator to the acoustic pressure in the mouthpiece $p(t)$. One of them describes the nonlinear excitation mechanism, while the other one describes the resonator, assumed to be linear, as follows:

$$u = u_{00} + Ap + Bp^2 + Cp^3 \tag{1}$$

$$u = h * p, \tag{2}$$

where $h(t)$ is the impulse response of the resonator, whose Fourier Transform is the input admittance $Y(\omega)$, the second equation often being written in the frequency domain:

$$U(\omega) = Y(\omega)P(\omega). \tag{3}$$

(The capital characters are used for the Fourier transform of the quantities of the time domain noted in small characters).

When no losses are considered, the input admittance is infinite for anti-resonance frequencies, and zero for resonance frequencies. If in addition the resonator is cylindrical and radiation entails only a length correction, the anti-resonances correspond to the even harmonics of the oscillation and the resonances to the odd harmonics. With these two conditions, the steady-state solution of the system is a pure square signal, the oscillation frequency being the classical value $c/4\ell$, where c is the speed of sound and ℓ the length of the resonator.

When losses are present, the maxima and minima become finite but the maxima remain large and the minima small as losses are rather weak. When dispersion is taken into account, or if there is a small deviation from the pure cylindrical shape of the resonator, the oscillation frequency, which is an unknown of the system, will in general be close to that of the nondispersive case, and the values of the admittance will not be strongly modified. Thus it is intuitive that a perturbation calculation is possible for both the shape of the signal and the oscillation frequency.

The first step of the reasoning is to avoid infinite or very large quantities, such as the impedance or admittance maxima in the frequency domain, or the maxima of the impulse response in the time domain. A solution consists in considering the frequency domain and dividing the harmonics into two classes: those with high impedance (low admittance) and those with low impedance (high admittance). The even harmonics of the pressure and the odd harmonics of the volume velocity are all small quantities, vanishing for the ideal condition of the pure Helmholtz motion, so they are good basic quantities for a perturbation calculation.

Note however that if a small shift in the oscillation frequency occurs because of dispersion or other causes, a perturbation calculation can become very bad for higher order harmonics. Thus it is essential to treat carefully the problem of the frequency shift. This will be explained in further detail in section 5. In the main part of the paper, we consider first that neither dispersion nor any other cause of frequency shift exist. As a consequence, the oscillation frequency is known and equal to $c/4\ell$. Moreover, in order to simplify the study of the problems of convergence for this preliminary study, the admittance is assumed to remain infinite for the even harmonics. The zeroth order solution, i.e. the pure Helmholtz motion, is recapitulated in section 2, then the first and second orders of the loss parameters are derived analytically in section 3. In section 4 the numerical treatment of the iterative procedure is presented as well as the results and a discussion about convergence.

2. ZEROTH ORDER SOLUTION: THE PURE HELMHOLTZ MOTION

The first step of the method is the separation of the even and odd harmonics of the signal, i.e. the symmetrical and antisymmetrical parts (with indices s and a, respectively). The two following equations are obtained (see Kergomard et al. [3]) and replace equation (1):

$$u_s = u_{00} + Ap_s + B(p_s^2 + p_a^2) + C(p_s^3 + 3p_sp_a^2) \tag{4}$$

$$u_a = Ap_a + 2Bp_sp_a + C(3p_ap_s^2 + p_a^3). \tag{5}$$

The equations replacing the linear equation (2) are obvious.

The even harmonics of the pressure are then ignored ($p_s = 0$), as explained earlier, and the initial set of equations is replaced by the following:

$$u_a = Ap_a + Cp_a^3 \tag{6}$$

and

$$u_a = h * p_a. \tag{7}$$

If no losses are present, since $h(t)$ only contains even harmonics and p_a only contains by definition the odd harmonics of p, equation (7) leads to $u_a = 0$. Then, eliminating in equation (6) the static solution $p_a = 0$, the pure Helmholtz motion is found to be

$$p_a = \pm\sqrt{-A/C} \tag{8}$$

where C is negative [2]. $A = 0$ at the threshold of oscillation. The Fourier components of this square-wave signal are well known. This is the zeroth order solution, and we will denote it p_0.

The volume velocity u_s is constant (only the d.c. component exists), and can be deduced by using equation (4).

3. FIRST AND SECOND ORDER SOLUTIONS

The basic principle of the method relies on the hypothesis that, for the components of $h(t)$ existing in the signals u_a and p_a, $h(t)$ is a first order function of a loss parameter. As an example, for pure cylindrical, sufficiently narrow tubes, $h(t)$ is proportional to the loss parameter η equal to $\sqrt{\ell\ell_v}/r$, where ℓ_v is the characteristic length of the viscous effects [2] and r the radius of the tube. Expanding the acoustic pressure at successive orders of η i.e. writing $p_a = p_0 + p_1 + p_2 + \ldots$, it is possible to solve equations (6) and (7). At the first order, the following result is obtained:

$$h * p_0 = Ap_1 + 3Cp_0^2p_1. \tag{9}$$

Using the zeroth order result (8), it can be simplified to

$$h * p_0 = -2Ap_1. \tag{10}$$

This result is extremely simple to write in the frequency domain since it is linear:

$$P_{1n} = -\frac{1}{2}\frac{Y_n}{A}P_{0n}, \tag{11}$$

the index n corresponding to the nth (odd) Fourier component of the signal. It is remarkable that the result for the first harmonic is identical to the result of the "first harmonic method," which consists in the approximation of the unknown signal by its first component only, as follows [2]:

$$P_1 = \frac{1}{3}\sqrt{\frac{Y_1 - A}{C}} = \frac{1}{3}\sqrt{-\frac{A}{C}}\left(1 - \frac{1}{2}\frac{Y_1}{A} + O\left(\frac{Y_1^2}{A^2}\right)\right). \tag{12}$$

The first term on the right-hand side is the value of the result when no losses exist. The extension of this approximation, called the "Variable Truncation Method" [3], also gives results for higher order harmonics, but these are different from the result (11), as the third harmonic, for example, depends on both Y_1 and Y_3. In any case, equation (12) allows an intuitive idea of the interest of the method presented in the present paper. The relevant quantities are the ratios Y_n/A: they are small either if losses are small or if the excitation level γ, directly related to A, is strong. Near the oscillation threshold, which satisfies exactly $A = Y_1$ (see Grand et al. [4]), the quantity Y_1/A approaches unity, and the series expansion (12) is thus not of interest.

A conclusion is that the present method is without interest near the oscillation threshold, and is complementary to the methods valid near the threshold, i.e. the Small Oscillation Method (see Worman [5] and reference [4]) or the more general Variable Truncation Method. We remind that these two methods are exact near the oscillation threshold under conditions related to the relative height of the first impedance peak and the other peaks. On the other hand, the present method is particularly well adapted when $Y_1/A \ll 1$. For example, when losses are neglected, the Harmonic Balance Method (see Farner et al. [6]) fails to converge toward the Helmholtz solution, which is given by order 0 of the present method.

Another feature concerning the present method is that the quantities Y_n/A can be relatively large for the higher order harmonics, since losses in general increase with frequency, but as convergence is obtained (see below), this must be explained by the small influence of the higher order harmonics on the result.

Continuing the procedure, the following nonlinear result is obtained for the second order of the perturbation:

$$h * p_1 = -2Ap_2 + 3Cp_0p_1^2. \tag{13}$$

The last term can be computed either in the time domain, using sampling, or in the Fourier domain, using the product of infinite series. The second method (with a truncation of the series to a certain number of harmonics) is used in the following because the left-hand side of equation (13) is easily calculated in the frequency domain. If the series are truncated to the first harmonic, the second-order result is obtained:

$$P_1 = P_{01}\left[1 - \frac{Y_1}{2A} + \frac{Y_1^2}{4A^2}\left(1 - \frac{18}{\pi^2}\right)\right] \tag{14}$$

which differs from that of the first harmonic method (equation (12)). Nevertheless, this equation can provide some useful analytical approximations even for a low order of the truncation.

4. HIGHER ORDER SOLUTIONS

Equation (13) shows the general shape of the higher order solutions:

$$h * p_{m-1} = -2Ap_m + C\times\begin{pmatrix}\text{products of three}\\\text{lower-order solutions}\end{pmatrix} \tag{15}$$

which need to be computed numerically. The algorithms are rather simple to establish and therefore not presented here. The product of three series is reduced to a double product for the calculation of the amplitude of the harmonics of the order m in the loss parameter.

The first property to study is the convergence for the order m when the number of harmonics increases. The modulus of a few odd harmonics of p, divided by their value at 100 harmonics, are plotted in figure 1 for order $m = 5$, $\gamma = 0.4$, and $\eta = 0.002$. It can be seen that for these parameter values (not too close to the oscillation threshold and for rather weak losses), the moduli are converging as the number of harmonics increase.

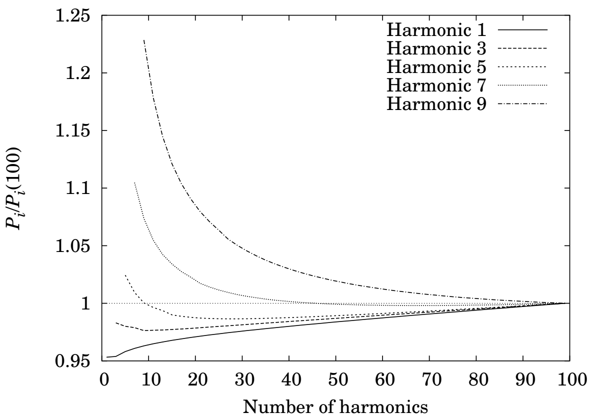

Figure 1: Modulus of the first five odd harmonics of p (order 5, $\gamma = 0.4$, $\eta = 0.002$) for an increasing number of harmonics, divided by their value at 100 harmonics

The second property to study is the convergence of the first harmonic P_1 as the order of η increases. As expected, the convergence is rather rapid for high excitation levels γ and for low losses. Figure 2 shows the first harmonic of the pressure as γ is varied, for different orders of approximation. Moreover, the solution given by the Harmonic Balance Method (HBM) is plotted for comparison. In this case, losses are rather weak ($\eta = 0.002$), and excepted in the vicinity of the oscillation threshold, few orders are required to converge toward the solution given by the HBM.

Results for the difficult case of rather strong losses ($\eta = 0.02$) are presented in figure 3. As expected, convergence toward the solution given by the HBM is more questionable near the threshold

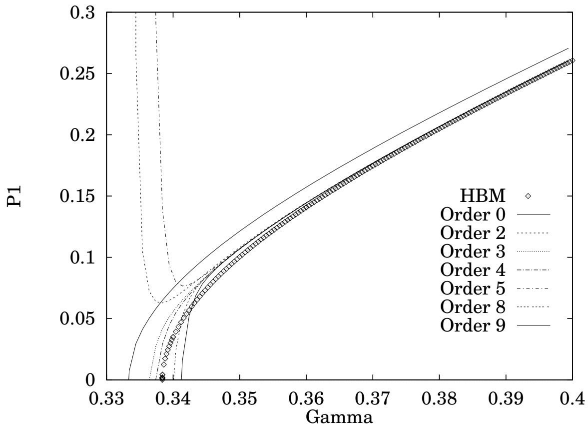

Figure 2: First harmonic of the pressure, when γ is varied, for different orders of approximation compared to the solution given by the HBM ($\eta = 0.002$, 50 harmonics)

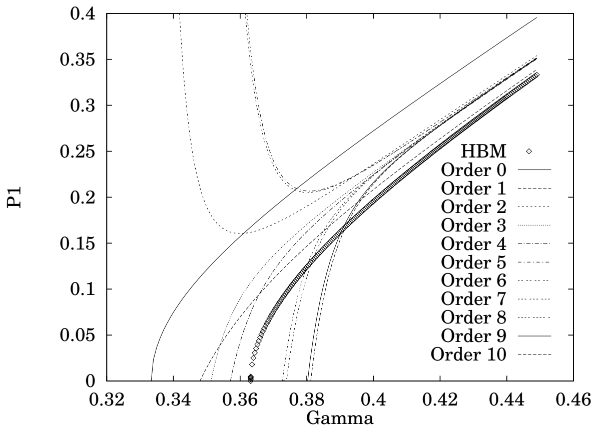

Figure 3: First harmonic of the pressure, when γ is varied, for different orders of approximation compared to the solution given by the HBM ($\eta = 0.02$, 50 harmonics)

of oscillation. More surprisingly, however, the converging results far from the threshold deviate from that of the HBM. This has to be further investigated.

In order to solve completely the initial set of equations (1) and (2), it is necessary to calculate the even harmonics, i.e. more generally the low-impedance components of the signal. This is done by a straightforward extension of the above-explained principle: expansion of the volume velocity u_s at successive orders of the loss parameters and solving of the set of equations (4) and (5).

5. HOW TO TREAT THE QUESTION OF THE PLAYING FREQUENCY?

Another important point is how to calculate the (unknown) oscillation frequency when for instance dispersion occurs. As explained earlier, it is absolutely necessary that, during each step of the iteration, the frequency does not change, otherwise the perturbation calculation is impossible (for higher order harmonics, the error would be unacceptable). Actually, the playing frequency can be determined by a parallel calculation. At the zeroth order, the frequency is known to be $c/4\ell$, and the solution for the amplitudes is known by equation (8). For the first and higher orders, one supplementary equation is necessary. We remark that the "reactive power" equation (see Boutillon and Gibiat [7]) is an excellent choice for such a supplementary equation:

$$\sum_{n=0}^{\infty} n\Im m(Y_n)\,|P_n|^2 = 0. \qquad (16)$$

(This is based on the fact that the nonlinear characteristic is independent of time, and is contained in the HBM equations.) After truncating it to the convenient number of harmonics, the frequency can be calculated in each step by noting the admittance Y_n for the nth harmonic depends on the frequency. Once the first order approximation of the frequency is determined, the first order approximation of the amplitudes is calculated using the admittance valid for the frequency just obtained. Compared to the simplified case studied in the previous sections, one supplementary step needs to be solved at each order of the iteration.

6. GENERAL COMMENTS AND CONCLUSION

The method presented in the paper, the perturbation of the Helmholtz motion, has shown to be efficient except near the oscillation threshold, where the signals involve very few harmonics. This is not a surprise as using a square signal as a starting point is not appropriate for that case so far from a square wave. We notice that most of the self-sustained oscillation instruments have a behaviour of inverse bifurcation, i.e. that the solutions near the oscillation threshold are unstable, thus uninteresting (see e.g. Dalmont et al. [8]). For instruments like a clarinet, having a direct bifurcation, other methods need to be used near the threshold.

It remains to take into account both the even harmonics and the variation of the oscillation frequency with the excitation level. The extension to conical instruments is in principle easy, by separating the small and large impedance components. Concerning bowed string instruments, a generalization is possible, if a nonlinear, time-independent characteristic exists.

The question of transients should be explored. At least for resonators with harmonically related resonances, it seems to be possible to extent the method.

Finally, a major interest for the current method lies in the possibility to get analytical approximations for such nonlinear systems, e.g. at the first or second orders. This may be exploited to study the effect of for instance nonlinear functions that are discontinuous or have a discontinuous derivative, like the violin.

7. REFERENCES

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