

RELAXATION TWO-PHASE FLOW MODELS AND THE SUBCHARACTERISTIC CONDITION

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The *subcharacteristic condition* for hyperbolic relaxation systems states that wave velocities of an equilibrium system cannot exceed the corresponding wave velocities of its relaxation system. This condition is central to the stability of hyperbolic relaxation systems, and is expected to hold for most such models describing natural phenomena.

In this paper, we study a hierarchy of two-phase flow models. We consider relaxation with respect to volume transfer, heat transfer and mass transfer. We formally verify that our relaxation processes are consistent with the first and second laws of thermodynamics, and present analytical expressions for the wave velocities for each model in the hierarchy. Through an appropriate choice of variables, we prove directly by sums-of-squares that for all relaxation processes considered, the subcharacteristic condition holds for any thermodynamically stable equation of state.

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1. Introduction

Two-phase pipe flow simulations have a number of important industrial applications, including nuclear reactor safety analysis,^{3,19} petroleum production^{4,9} and CO₂ capture and storage.^{11,12} In order to obtain models sufficiently tractable for such large-scale industrial simulations, some simplifying assumptions must be

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made. In particular, most relevant flow models are averaged in space to yield one-dimensional systems of hyperbolic balance laws, expressible in the form:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \mathbf{B} \frac{\partial \mathbf{W}(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U}), \quad (1.1)$$

to be solved for the unknown M -vector \mathbf{U} .

Furthermore, dynamical two-phase flow processes will generally not take place in thermodynamic equilibrium. However, the relaxation time towards equilibrium may for several practical purposes be small. For such cases, the equilibrium assumption may be a valid approximation.

In this paper, we are interested in studying how the assumptions of mechanical, thermal, and phase equilibrium influence the propagation of pressure waves in the resulting fluid-mechanical models.

For smooth solutions, the models may be written in the general *relaxation* form¹⁴:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} = \frac{1}{\varepsilon} \mathbf{Q}(\mathbf{U}), \quad (1.2)$$

where

$$\mathbf{A} = \frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}} + \mathbf{B} \frac{\partial \mathbf{W}(\mathbf{U})}{\partial \mathbf{U}}. \quad (1.3)$$

The relaxation term \mathbf{Q} is assumed to be endowed with a constant linear operator \mathbf{P} with rank $m < M$ such that

$$\mathbf{P}\mathbf{Q}(\mathbf{U}) = 0. \quad (1.4)$$

This yields m homogeneous equations

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{P}\mathbf{A} \frac{\partial \mathbf{U}}{\partial x} = 0 \quad (1.5)$$

in the reduced variable

$$\mathbf{V} = \mathbf{P}\mathbf{U}. \quad (1.6)$$

We further assume that each \mathbf{V} uniquely determines a local equilibrium value $\mathbf{U} = \mathcal{E}(\mathbf{V})$, satisfying $\mathbf{Q}(\mathcal{E}(\mathbf{V})) = 0$ as well as

$$\mathbf{P}\mathcal{E}(\mathbf{V}) = \mathbf{V}. \quad (1.7)$$

One may then close the system (1.5) by imposing the equilibrium condition for \mathbf{U} :

$$\mathbf{U} = \mathcal{E}(\mathbf{V}). \quad (1.8)$$

As $\varepsilon \rightarrow 0$, the solutions to the relaxation system (1.2) are expected to approach the solutions to the relaxed system (1.5).

1.1. The subcharacteristic condition

In fluid mechanics, it has long been folklore knowledge that imposed equilibrium conditions tend to decrease the wave velocities predicted by the model. This has commonly been expressed as the “frozen” speed of sound being larger than the equilibrium speed of sound.^{7,15}

A mathematical understanding of this phenomenon was achieved through the concept of the *subcharacteristic condition*, first introduced by Liu.¹⁰ A general, precise definition is stated by Chen et al.⁵:

Definition 1. Let the M eigenvalues of the relaxing system (1.2) be given by

$$\lambda_1 \leq \dots \leq \lambda_k \leq \lambda_{k+1} \leq \dots \leq \lambda_M \quad (1.9)$$

and the m eigenvalues of the relaxed system (1.5) and (1.8) be given by

$$\tilde{\lambda}_1 \leq \dots \leq \tilde{\lambda}_j \leq \tilde{\lambda}_{j+1} \leq \dots \leq \tilde{\lambda}_m. \quad (1.10)$$

Herein, the relaxation system (1.2) is applied to a local equilibrium state $\mathbf{U} = \mathcal{E}(\mathbf{V})$ such that

$$\lambda_k = \lambda_k(\mathcal{E}(\mathbf{V})), \quad \tilde{\lambda}_j = \tilde{\lambda}_j(\mathbf{V}). \quad (1.11)$$

Now let the $\tilde{\lambda}_j$ be *interlaced* with λ_k in the following sense: Each $\tilde{\lambda}_j$ lies in the closed interval $[\lambda_j, \lambda_{j+M-m}]$. Then the relaxed system (1.5) is said to satisfy the **subcharacteristic condition** with respect to (1.2).

The subcharacteristic condition is central to the question of stability of the stiff limit of relaxation systems. As pointed out by Natalini,¹⁴ the condition can be interpreted as a causality principle. Source terms have only a local influence on the system, and can therefore not increase the characteristic speeds of information. We expect this to hold also in the stiff limit $\varepsilon \rightarrow 0$.

Chen et al.⁵ proved that if the relaxation system (1.2) may be equipped with a convex entropy function that is dissipated by the relaxation term, then the subcharacteristic condition holds. Furthermore, a converse holds for linear systems and general 2×2 systems.

1.2. Outline of this paper

In recent years, there has been significant interest in various relaxation models for two-phase flows.^{1,6,13,16,17,20} Much of this work builds on the classic paper by Baer and Nunziato.² Few of these recent works seem to focus on the subcharacteristic condition, an observation which motivates our current paper. We here wish to systematically investigate various two-phase flow relaxation models, with a particular emphasis on the interlacing of eigenvalues as described in Definition 1.

The relaxation models we consider are highly related to the models presented in recent works by Saurel et al.¹⁷ and Zein et al.²⁰ Furthermore, exact expressions for the eigenvalues of several of our models already abound in the literature.^{6,8,17,18}

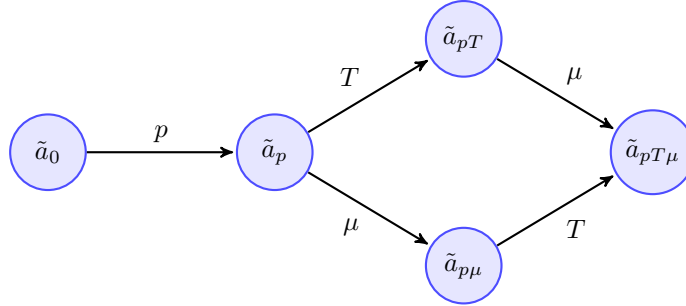


Fig. 1. A hierarchy of relaxation models for two-phase flows. The circles represent the various models, and the arrows represent relaxation processes.

Combining these established results with some original analysis, we will show directly that all our relaxation processes satisfy the subcharacteristic condition.

First, we focus on formulating explicit relaxation procedures that formally respect the first and second laws of thermodynamics. Then, we make our main original contribution, by expressing the eigenstructure of the models in natural variables that allows us to prove the subcharacteristic conditions directly by sums-of-squares. In particular, for any model \mathcal{B} that arises from a model \mathcal{A} through a relaxation procedure, we show that the mixture sound velocities \tilde{a} are related by

$$\tilde{a}_{\mathcal{B}}^{-2} = \tilde{a}_{\mathcal{A}}^{-2} + Z_{\mathcal{B}}^{\mathcal{A}}, \quad (1.12)$$

where $Z_{\mathcal{B}}^{\mathcal{A}}$ involves only sums-of-squares with positive coefficients. From this, the subcharacteristic conditions follow directly.

The relaxation procedures we will consider are the following:

- (i) *Mechanical relaxation*, or relaxation with respect to volume transfer. As this drives the models towards pressure equilibrium, we will denote this as ***p*-relaxation**.
- (ii) *Thermal relaxation*, or relaxation with respect to heat transfer. As this drives the models towards temperature equilibrium, we will denote this as ***T*-relaxation**.
- (iii) *Material relaxation*, or relaxation with respect to phase transfer. As this drives the models towards chemical-potential equilibrium, we will denote this as ***μ*-relaxation**.

For all the relaxation models, we will make the *a priori* assumption of dynamic equilibrium; the two phases are assumed to flow with the same velocity. The hierarchy of relaxation processes investigated in this paper is schematically presented in Figure 1.

Our paper is organized as follows: In Section 2, we present the basic relaxation model where heat, mass and volume transfer between the phases are all modelled by

relaxation source terms. We provide explicit expressions for these relaxation source terms that satisfy the first and second laws of thermodynamics, and derive the wave velocities of this model.

Then, in Section 3, we consider the model that arises from performing the p -relaxation on the basic model. In Section 4, we consider pT -relaxation; the simultaneous relaxation of both the pressures and the temperatures. In Section 5, we consider simultaneous relaxation of the pressures and chemical potentials, here denoted as $p\mu$ -relaxation. For both these cases we prove the subcharacteristic condition in the sense of Definition 1.

In Section 6, we consider the fully relaxed model, also denoted as the *homogeneous equilibrium* (HEM) model. We review the wave structure of this model, and recover the well-known fact that this model has a discontinuous mixture sound velocity in the limit where one of the phases disappears. Remarkably, it turns out that this discontinuous behaviour cannot be attributed to one single relaxation procedure; each procedure in itself yields continuous behaviour. Rather, the discontinuity of the HEM model is an emergent phenomenon, arising only when all relaxation procedures are simultaneously applied.

In Section 7, we illustrate our results by plotting the various sound velocities for a couple of practically relevant two-phase mixtures. Finally, in Section 8, we briefly discuss and summarize our results.

2. The Basic Model

In this section, we describe our fundamental model where no thermodynamic equilibrium assumptions are made. In particular, we assume that both the gas (g) and liquid phase (ℓ) have separate pressures, temperatures and chemical potentials. However, we assume that both phases flow with the common velocity v . Heat, mass and volume transfer are now modelled by relaxation source terms.

2.1. Mass conservation

Without loss of generality, we may now formulate mass conservation equations as follows:

$$\frac{\partial}{\partial t}(\rho_g \alpha_g) + \frac{\partial}{\partial x}(\rho_g \alpha_g v) = \mathcal{K}(\mu_\ell - \mu_g), \quad (2.1)$$

$$\frac{\partial}{\partial t}(\rho_\ell \alpha_\ell) + \frac{\partial}{\partial x}(\rho_\ell \alpha_\ell v) = \mathcal{K}(\mu_g - \mu_\ell), \quad (2.2)$$

where we have used the following nomenclature:

ρ_k	- density of phase k	(kg/m ³),
v	- velocity common to both phases	(m/s),
α_k	- volume fraction of phase k	($\alpha_g + \alpha_\ell = 1$),
μ_k	- chemical potential of phase k	m ² /s ² ,
\mathcal{K}	- phase transfer relaxation coefficient	kg·s/m ⁵ .

Here we only assume that $\mathcal{K} \geq 0$. Now, these equations have the following properties:

- total mass is conserved;
- mass flows from high to low chemical potential;
- there is no phase transfer when the chemical potentials are equal.

2.2. *Volume advection*

We now assume that in Lagrangian coordinates, only pressure differences will induce a volume transfer between the phases. In this respect, we follow in the footsteps of classical two-phase relaxation models.^{2,16}

$$\frac{\partial \alpha_g}{\partial t} + v \frac{\partial \alpha_g}{\partial x} = \mathcal{J}(p_g - p_\ell). \quad (2.3)$$

Herein, a couple of new symbols are introduced:

$$\begin{aligned} p_k & - \text{pressure of phase } k && \text{Pa,} \\ \mathcal{J} & - \text{volume transfer relaxation coefficient} && (\text{Pa}\cdot\text{s})^{-1}, \end{aligned}$$

where we assume that $\mathcal{J} \geq 0$.

2.3. *Momentum conservation*

The common velocity v between the phases may be obtained from conservation of total momentum:

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + \alpha_g p_g + \alpha_\ell p_\ell) = 0, \quad (2.4)$$

where we have used the shorthand

$$\rho = \rho_g \alpha_g + \rho_\ell \alpha_\ell. \quad (2.5)$$

2.4. *Energy balance*

The energy balance equations will consist of two parts:

- (i) the *transport* part, involving spatial derivatives;
- (ii) the *relaxation* part, involving source terms.

We will find it instructive to focus on these parts separately when constructing our energy balance equations.

2.4.1. *Energy relaxation terms*

We assume that mass, heat and volume transfer all contribute to energy transfer between the phases. If we for the moment ignore transport effects, this can be

written with no loss of generality as ordinary differential equations:

$$\frac{d(\rho_g \alpha_g e_g)}{dt} = \mathcal{H}(T_\ell - T_g) + p^* \mathcal{J}(p_\ell - p_g) + \mu^* \mathcal{K}(\mu_\ell - \mu_g), \quad (2.6)$$

$$\frac{d(\rho_\ell \alpha_\ell e_\ell)}{dt} = \mathcal{H}(T_g - T_\ell) + p^* \mathcal{J}(p_g - p_\ell) + \mu^* \mathcal{K}(\mu_g - \mu_\ell), \quad (2.7)$$

where we have assumed only that all relaxation processes individually conserve total energy, i.e.

$$\frac{d(\rho_g \alpha_g e_g + \rho_\ell \alpha_\ell e_\ell)}{dt} = 0. \quad (2.8)$$

Herein, we have introduced some new symbols:

e_k	- specific internal energy of phase k	m^2/s^2 ,
\mathcal{H}	- heat transfer relaxation coefficient	$\text{kg}/(\text{m}\cdot\text{s}^3\cdot\text{K})$,
p^*	- effective pressure at the gas-liquid interface	Pa,
μ^*	- effective chemical potential at the gas-liquid interface	m^2/s^2 .

2.4.2. Entropy source terms

As our next step, we aim to transform these energy source terms into corresponding source terms for the entropy. Ignoring transport effects in (2.1)–(2.3), we may write

$$\frac{d(\rho_g \alpha_g)}{dt} = \mathcal{K}(\mu_\ell - \mu_g), \quad (2.9)$$

$$\frac{d(\rho_\ell \alpha_\ell)}{dt} = \mathcal{K}(\mu_g - \mu_\ell), \quad (2.10)$$

$$\frac{d\alpha_g}{dt} = \mathcal{J}(p_g - p_\ell). \quad (2.11)$$

Using these results together with (2.6)–(2.7), we obtain

$$\rho_g \alpha_g T_g \frac{ds_g}{dt} = \mathcal{H}(T_\ell - T_g) + (p^* - p_g) \mathcal{J}(p_\ell - p_g) + (\mu^* - \mu_g - T_g s_g) \mathcal{K}(\mu_\ell - \mu_g), \quad (2.12)$$

$$\rho_\ell \alpha_\ell T_\ell \frac{ds_\ell}{dt} = \mathcal{H}(T_g - T_\ell) + (p^* - p_\ell) \mathcal{J}(p_g - p_\ell) + (\mu^* - \mu_\ell - T_\ell s_\ell) \mathcal{K}(\mu_g - \mu_\ell), \quad (2.13)$$

where we have used the fundamental thermodynamic differential

$$de_k = T_k ds_k + \frac{p}{\rho_k^2} d\rho_k, \quad k \in \{g, \ell\} \quad (2.14)$$

as well as the product rule for derivatives. Herein, s_k is the specific entropy of phase k .

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2.4.3. Entropy evolution equations

We now make the assumption that in Lagrangian coordinates, only the relaxation source terms contribute to entropy changes. More precisely, we introduce the *material* derivative

$$D_t = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \quad (2.15)$$

and rewrite (2.12)–(2.13) as

$$\rho_g \alpha_g T_g D_t s_g = \mathcal{H}(T_\ell - T_g) + (p^* - p_g) \mathcal{J}(p_\ell - p_g) + (\mu^* - \mu_g - T_g s_g) \mathcal{K}(\mu_\ell - \mu_g), \quad (2.16)$$

$$\rho_\ell \alpha_\ell T_\ell D_t s_\ell = \mathcal{H}(T_g - T_\ell) + (p^* - p_\ell) \mathcal{J}(p_g - p_\ell) + (\mu^* - \mu_\ell - T_\ell s_\ell) \mathcal{K}(\mu_g - \mu_\ell). \quad (2.17)$$

2.4.4. The second law of thermodynamics

By using (2.1)–(2.2), we may rewrite (2.16)–(2.17) as

$$\begin{aligned} T_g \left(\frac{\partial}{\partial t} (\rho_g \alpha_g s_g) + \frac{\partial}{\partial x} (\rho_g \alpha_g s_g v) \right) \\ = \mathcal{H}(T_\ell - T_g) + (p^* - p_g) \mathcal{J}(p_\ell - p_g) + (\mu^* - \mu_g) \mathcal{K}(\mu_\ell - \mu_g), \end{aligned} \quad (2.18)$$

$$\begin{aligned} T_\ell \left(\frac{\partial}{\partial t} (\rho_\ell \alpha_\ell s_\ell) + \frac{\partial}{\partial x} (\rho_\ell \alpha_\ell s_\ell v) \right) \\ = \mathcal{H}(T_g - T_\ell) + (p^* - p_\ell) \mathcal{J}(p_g - p_\ell) + (\mu^* - \mu_\ell) \mathcal{K}(\mu_g - \mu_\ell). \end{aligned} \quad (2.19)$$

Proposition 1. *Sufficient conditions for the relaxation model given by (2.1)–(2.4), (2.16)–(2.17) to satisfy the second law of thermodynamics are:*

$$\mathcal{H} \geq 0, \quad (2.20)$$

$$\mathcal{J} \geq 0, \quad (2.21)$$

$$\mathcal{K} \geq 0, \quad (2.22)$$

$$\min(p_g, p_\ell) \leq p^* \leq \max(p_g, p_\ell), \quad (2.23)$$

$$\min(\mu_g, \mu_\ell) \leq \mu^* \leq \max(\mu_g, \mu_\ell). \quad (2.24)$$

Proof. We define the total entropy S through

$$S = \rho_g \alpha_g s_g + \rho_\ell \alpha_\ell s_\ell. \quad (2.25)$$

Furthermore, we write (2.23)–(2.24) as

$$p^* = \beta_p p_g + (1 - \beta_p) p_\ell, \quad \beta_p \in [0, 1], \quad (2.26)$$

$$\mu^* = \beta_\mu \mu_g + (1 - \beta_\mu) \mu_\ell, \quad \beta_\mu \in [0, 1]. \quad (2.27)$$

Then from (2.18)–(2.19) we obtain

$$\begin{aligned} \frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(Sv) &= \mathcal{H} \frac{(T_\ell - T_g)^2}{T_g T_\ell} \\ &+ \left(\frac{1 - \beta_p}{T_g} + \frac{\beta_p}{T_\ell} \right) \mathcal{J}(p_\ell - p_g)^2 + \left(\frac{1 - \beta_\mu}{T_g} + \frac{\beta_\mu}{T_\ell} \right) \mathcal{K}(\mu_\ell - \mu_g)^2. \end{aligned} \quad (2.28)$$

Now inside a *closed* region R the global entropy Ω is given by

$$\Omega(t) = \int_R S(x, t) dx. \quad (2.29)$$

Then from (2.28) we obtain

$$\begin{aligned} \frac{d\Omega}{dt} &= \int_R \left(\mathcal{H} \frac{(T_\ell - T_g)^2}{T_g T_\ell} + \left(\frac{1 - \beta_p}{T_g} + \frac{\beta_p}{T_\ell} \right) \mathcal{J}(p_\ell - p_g)^2 \right) dx \\ &+ \int_R \left(\left(\frac{1 - \beta_\mu}{T_g} + \frac{\beta_\mu}{T_\ell} \right) \mathcal{K}(\mu_\ell - \mu_g)^2 \right) dx. \end{aligned} \quad (2.30)$$

Assuming positive temperatures, we observe that all terms under the integral are non-negative. Hence

$$\frac{d\Omega}{dt} \geq 0, \quad (2.31)$$

in accordance with the second law. \square

2.4.5. Internal energy evolution equations

By writing the fundamental thermodynamic differential as

$$D_t e_k = T_k D_t s_k + \frac{p_k}{\rho_k^2} D_t \rho_k, \quad k \in \{g, \ell\}, \quad (2.32)$$

we may equivalently express the entropy equations (2.16)–(2.17) as internal energy evolution equations:

$$\frac{\partial}{\partial t}(\rho_g \alpha_g e_g) + \frac{\partial}{\partial x}(\rho_g \alpha_g e_g v) + \alpha_g p_g \frac{\partial v}{\partial x} = \mathcal{H}(T_\ell - T_g) + p^* \mathcal{J}(p_\ell - p_g) + \mu^* \mathcal{K}(\mu_\ell - \mu_g) \quad (2.33)$$

and

$$\frac{\partial}{\partial t}(\rho_\ell \alpha_\ell e_\ell) + \frac{\partial}{\partial x}(\rho_\ell \alpha_\ell e_\ell v) + \alpha_\ell p_\ell \frac{\partial v}{\partial x} = \mathcal{H}(T_g - T_\ell) + p^* \mathcal{J}(p_g - p_\ell) + \mu^* \mathcal{K}(\mu_g - \mu_\ell). \quad (2.34)$$

This result now follows from (2.1)–(2.2), (2.3) and (2.16)–(2.17) by expanding and collecting derivatives.

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2.4.6. Kinetic energy evolution equations

Based on momentum conservation, kinetic energy evolution equations can also be derived. In particular, by using conservation of total mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \quad (2.35)$$

we may write (2.4) as

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial x}(\alpha_g p_g + \alpha_\ell p_\ell) = 0. \quad (2.36)$$

By collecting derivatives in (2.1), (2.2) and (2.36), we now obtain

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_g \alpha_g v^2 \right) + \frac{\partial}{\partial x} \left(\frac{1}{2} \rho_g \alpha_g v^3 \right) + v \frac{\partial}{\partial x}(\alpha_g p_g) \\ + \frac{v}{\rho} \left(\rho_g \alpha_g \frac{\partial}{\partial x}(\alpha_\ell p_\ell) - \rho_\ell \alpha_\ell \frac{\partial}{\partial x}(\alpha_g p_g) \right) = \frac{1}{2} v^2 \mathcal{K}(\mu_\ell - \mu_g), \end{aligned} \quad (2.37)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_\ell \alpha_\ell v^2 \right) + \frac{\partial}{\partial x} \left(\frac{1}{2} \rho_\ell \alpha_\ell v^3 \right) + v \frac{\partial}{\partial x}(\alpha_\ell p_\ell) \\ + \frac{v}{\rho} \left(\rho_\ell \alpha_\ell \frac{\partial}{\partial x}(\alpha_g p_g) - \rho_g \alpha_g \frac{\partial}{\partial x}(\alpha_\ell p_\ell) \right) = \frac{1}{2} v^2 \mathcal{K}(\mu_g - \mu_\ell). \end{aligned} \quad (2.38)$$

2.4.7. The first law of thermodynamics

We now have sufficient results to show that our relaxation procedures respect the first law of thermodynamics.

Proposition 2. *The basic relaxation model described in this section conserves total energy; in particular,*

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(v(E + \alpha_g p_g + \alpha_\ell p_\ell)) = 0, \quad (2.39)$$

where

$$E = E_g + E_\ell, \quad (2.40)$$

$$E_k = \rho_k \alpha_k \left(e_k + \frac{1}{2} v_k^2 \right), \quad k \in \{g, \ell\}. \quad (2.41)$$

Proof. Add (2.33) to (2.37), as well as (2.34) to (2.38), to obtain energy evolution equations for each phase:

$$\begin{aligned} \frac{\partial E_g}{\partial t} + \frac{\partial}{\partial x}(v(E_g + \alpha_g p_g)) + \frac{v}{\rho} \left(\rho_g \alpha_g \frac{\partial}{\partial x}(\alpha_\ell p_\ell) - \rho_\ell \alpha_\ell \frac{\partial}{\partial x}(\alpha_g p_g) \right) \\ = \mathcal{H}(T_\ell - T_g) + p^* \mathcal{J}(p_\ell - p_g) + \left(\mu^* + \frac{1}{2} v^2 \right) \mathcal{K}(\mu_\ell - \mu_g), \end{aligned} \quad (2.42)$$

$$\begin{aligned} \frac{\partial E_\ell}{\partial t} + \frac{\partial}{\partial x} (v(E_\ell + \alpha_\ell p_\ell)) + \frac{v}{\rho} \left(\rho_\ell \alpha_\ell \frac{\partial}{\partial x} (\alpha_g p_g) - \rho_g \alpha_g \frac{\partial}{\partial x} (\alpha_\ell p_\ell) \right) \\ = \mathcal{H}(T_g - T_\ell) + p^* \mathcal{J}(p_g - p_\ell) + \left(\mu^* + \frac{1}{2} v^2 \right) \mathcal{K}(\mu_g - \mu_\ell). \end{aligned} \quad (2.43)$$

Add (2.42) and (2.43) to obtain the result. \square

2.5. Wave velocities

We have now established a basic relaxation model in a rather general form, where the system is driven towards equilibria characterized by the pressures, temperatures and chemical potentials being equal. In this section, we aim to derive the wave velocities in the non-stiff limit of this model. To this end, we will find it convenient to express the model in the following form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0, \quad (2.44)$$

$$D_t Y = \frac{\mathcal{K}}{\rho} (\mu_\ell - \mu_g), \quad (2.45)$$

$$D_t \alpha_g = \mathcal{J}(p_g - p_\ell), \quad (2.46)$$

$$\frac{\partial}{\partial t} (\rho v) + \frac{\partial}{\partial x} (\rho v^2 + \alpha_g p_g + \alpha_\ell p_\ell) = 0, \quad (2.47)$$

$$D_t s_g = \frac{\mathcal{H}}{\rho_g \alpha_g} \frac{T_\ell - T_g}{T_g} + \frac{p^* - p_g}{\rho_g \alpha_g T_g} \mathcal{J}(p_\ell - p_g) + \left(\frac{\mu^* - \mu_g}{T_g} - s_g \right) \frac{\mathcal{K}}{\rho_g \alpha_g} (\mu_\ell - \mu_g), \quad (2.48)$$

$$D_t s_\ell = \frac{\mathcal{H}}{\rho_\ell \alpha_\ell} \frac{T_g - T_\ell}{T_\ell} + \frac{p^* - p_\ell}{\rho_\ell \alpha_\ell T_\ell} \mathcal{J}(p_g - p_\ell) + \left(\frac{\mu^* - \mu_\ell}{T_\ell} - s_\ell \right) \frac{\mathcal{K}}{\rho_\ell \alpha_\ell} (\mu_g - \mu_\ell), \quad (2.49)$$

where

$$Y = \frac{\rho_g \alpha_g}{\rho}. \quad (2.50)$$

Herein, (2.45) is obtained by writing (2.1) as

$$\frac{\partial}{\partial t} (\rho Y) + \frac{\partial}{\partial x} (\rho Y v) = \mathcal{K}(\mu_\ell - \mu_g) \quad (2.51)$$

and using (2.44).

Proposition 3. *The vector of eigenvalues of the basic relaxation model (2.44)–(2.49) is given by*

$$\Lambda_0 = \begin{bmatrix} v - \tilde{a}_0 \\ v \\ v \\ v \\ v \\ v + \tilde{a}_0 \end{bmatrix}, \quad (2.52)$$

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where

$$\tilde{a}_0^2 = Yc_g^2 + (1 - Y)c_\ell^2, \quad (2.53)$$

$$c_k = \left(\frac{\partial p}{\partial \rho_k} \right)_{s_k}, \quad k \in \{g, \ell\}. \quad (2.54)$$

Proof. From (2.45)–(2.46) and (2.48)–(2.49), we see directly that $(Y, \alpha_g, s_g, s_\ell)$ are characteristic variables corresponding to an eigenvalue with magnitude v . Now if we set $dY = d\alpha_g = ds_g = ds_\ell = 0$, we obtain the following reduced model:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \quad (2.55)$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + \alpha_g p_g + \alpha_\ell p_\ell) = 0. \quad (2.56)$$

This can be written in quasilinear form:

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho v \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ \tilde{a}_0^2 - v^2 & 2v \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \rho \\ \rho v \end{bmatrix} = 0, \quad (2.57)$$

where

$$\tilde{a}_0^2 = \left(\frac{\partial}{\partial \rho}(\alpha_g p_g + \alpha_\ell p_\ell) \right)_{Y, \alpha_g, s_g, s_\ell}. \quad (2.58)$$

Hence the two missing waves of the full model are sound waves with velocities $v \pm \tilde{a}_0$. It now remains to prove that (2.53) and (2.58) are equivalent. The assumption of constant entropies gives us

$$dp_g = c_g^2 d\rho_g, \quad (2.59)$$

$$dp_\ell = c_\ell^2 d\rho_\ell. \quad (2.60)$$

Furthermore, the assumption of constant volume and mass fractions gives us

$$dY = \frac{\alpha_g}{\rho} \left(d\rho_g - \frac{\rho_g}{\rho} d\rho \right) = -\frac{\alpha_\ell}{\rho} \left(d\rho_\ell - \frac{\rho_\ell}{\rho} d\rho \right) = 0, \quad (2.61)$$

Hence

$$d\rho = \frac{\rho}{\rho_g} d\rho_g = \frac{\rho}{\rho_\ell} d\rho_\ell. \quad (2.62)$$

We now recover (2.53) by using (2.59)–(2.60) and (2.62) in (2.58). \square

3. Pressure Relaxation

In this section, we investigate the effect of p -relaxation in the basic model described in Section 2. We expect the limit $\mathcal{J} \rightarrow \infty$ to correspond to replacing (2.3) with the assumption

$$p_g = p_\ell = p. \quad (3.1)$$

Then from (2.23) we obtain:

$$p^* = p. \quad (3.2)$$

Now the energy equations (2.42)–(2.43) can be rewritten as

$$\begin{aligned} \frac{\partial E_g}{\partial t} + \frac{\partial}{\partial x} (vE_g) + \frac{\rho_g \alpha_g}{\rho} v \frac{\partial p}{\partial x} + p \left(\frac{\partial \alpha_g}{\partial t} + \frac{\partial}{\partial x} (\alpha_g v) \right) \\ = \mathcal{H}(T_\ell - T_g) + \left(\mu^* + \frac{1}{2} v^2 \right) \mathcal{K}(\mu_\ell - \mu_g), \end{aligned} \quad (3.3)$$

$$\begin{aligned} \frac{\partial E_\ell}{\partial t} + \frac{\partial}{\partial x} (vE_\ell) + \frac{\rho_\ell \alpha_\ell}{\rho} v \frac{\partial p}{\partial x} + p \left(\frac{\partial \alpha_\ell}{\partial t} + \frac{\partial}{\partial x} (\alpha_\ell v) \right) \\ = \mathcal{H}(T_g - T_\ell) + \left(\mu^* + \frac{1}{2} v^2 \right) \mathcal{K}(\mu_g - \mu_\ell). \end{aligned} \quad (3.4)$$

Furthermore, following the approach detailed in Ref. 6, we can derive volume fraction evolution equations:

$$\begin{aligned} \frac{\partial \alpha_g}{\partial t} + \frac{\partial}{\partial x} (\alpha_g v) - \alpha_g \frac{\rho \tilde{a}_p^2}{\rho_g c_g^2} \frac{\partial v}{\partial x} = \frac{\rho \tilde{a}_p^2}{\rho_g c_g^2 \rho_\ell c_\ell^2} (\alpha_g \Gamma_\ell + \alpha_\ell \Gamma_g) \mathcal{H}(T_\ell - T_g) \\ + \frac{\rho \tilde{a}_p^2}{\rho_g c_g^2 \rho_\ell c_\ell^2} (\alpha_g (\Gamma_\ell \mu^* - \Gamma_\ell h_\ell + c_\ell^2) + \alpha_\ell (\Gamma_g \mu^* - \Gamma_g h_g + c_g^2)) \mathcal{K}(\mu_\ell - \mu_g) \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \frac{\partial \alpha_\ell}{\partial t} + \frac{\partial}{\partial x} (\alpha_\ell v) - \alpha_\ell \frac{\rho \tilde{a}_p^2}{\rho_\ell c_\ell^2} \frac{\partial v}{\partial x} = \frac{\rho \tilde{a}_p^2}{\rho_g c_g^2 \rho_\ell c_\ell^2} (\alpha_g \Gamma_\ell + \alpha_\ell \Gamma_g) \mathcal{H}(T_g - T_\ell) \\ + \frac{\rho \tilde{a}_p^2}{\rho_g c_g^2 \rho_\ell c_\ell^2} (\alpha_g (\Gamma_\ell \mu^* - \Gamma_\ell h_\ell + c_\ell^2) + \alpha_\ell (\Gamma_g \mu^* - \Gamma_g h_g + c_g^2)) \mathcal{K}(\mu_g - \mu_\ell), \end{aligned} \quad (3.6)$$

where

$$h_k = e_k + \frac{p}{\rho_k} \quad \forall k \in \{g, \ell\}, \quad (3.7)$$

Γ_k is the *Grüneisen coefficient*:

$$\Gamma_k = \frac{1}{\rho_k} \left(\frac{\partial p}{\partial e_k} \right)_{\rho_k}, \quad (3.8)$$

and

$$\tilde{a}_p^{-2} = \rho \left(\frac{\alpha_g}{\rho_g c_g^2} + \frac{\alpha_\ell}{\rho_\ell c_\ell^2} \right). \quad (3.9)$$

3.1. The p -relaxed model

With respect to volume transfer, the relaxed model may now be written out in full as follows:

- *Mass conservation:*

$$\frac{\partial}{\partial t}(\rho_g \alpha_g) + \frac{\partial}{\partial x}(\rho_g \alpha_g v) = \mathcal{K}(\mu_\ell - \mu_g), \quad (3.10)$$

$$\frac{\partial}{\partial t}(\rho_\ell \alpha_\ell) + \frac{\partial}{\partial x}(\rho_\ell \alpha_\ell v) = \mathcal{K}(\mu_g - \mu_\ell). \quad (3.11)$$

- *Momentum conservation:*

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + p) = 0. \quad (3.12)$$

- *Energy balance:*

$$\begin{aligned} & \frac{\partial E_g}{\partial t} + \frac{\partial}{\partial x}(v E_g) + \frac{\rho_g \alpha_g}{\rho} v \frac{\partial p}{\partial x} + p \alpha_g \frac{\rho \tilde{a}_p^2}{\rho_g c_g^2} \frac{\partial v}{\partial x} \\ & = \left(1 - p \frac{\alpha_g \Gamma_\ell + \alpha_\ell \Gamma_g}{\rho_g \alpha_\ell c_g^2 + \rho_\ell \alpha_g c_\ell^2} \right) \mathcal{H}(T_\ell - T_g) \\ & + \left(\mu^* + \frac{1}{2} v^2 - p \frac{\alpha_g (\Gamma_\ell \mu^* - \Gamma_\ell h_\ell + c_\ell^2) + \alpha_\ell (\Gamma_g \mu^* - \Gamma_g h_g + c_g^2)}{\rho_g \alpha_\ell c_g^2 + \rho_\ell \alpha_g c_\ell^2} \right) \mathcal{K}(\mu_\ell - \mu_g), \end{aligned} \quad (3.13)$$

$$\begin{aligned} & \frac{\partial E_\ell}{\partial t} + \frac{\partial}{\partial x}(v E_\ell) + \frac{\rho_\ell \alpha_\ell}{\rho} v \frac{\partial p}{\partial x} + p \alpha_\ell \frac{\rho \tilde{a}_p^2}{\rho_\ell c_\ell^2} \frac{\partial v}{\partial x} \\ & = \left(1 - p \frac{\alpha_g \Gamma_\ell + \alpha_\ell \Gamma_g}{\rho_g \alpha_\ell c_g^2 + \rho_\ell \alpha_g c_\ell^2} \right) \mathcal{H}(T_g - T_\ell) \\ & + \left(\mu^* + \frac{1}{2} v^2 - p \frac{\alpha_g (\Gamma_\ell \mu^* - \Gamma_\ell h_\ell + c_\ell^2) + \alpha_\ell (\Gamma_g \mu^* - \Gamma_g h_g + c_g^2)}{\rho_g \alpha_\ell c_g^2 + \rho_\ell \alpha_g c_\ell^2} \right) \mathcal{K}(\mu_g - \mu_\ell). \end{aligned} \quad (3.14)$$

Herein, we obtain (3.13) and (3.14) by substituting (3.5)–(3.6) into (3.3)–(3.4).

3.2. Wave velocities

This model has been extensively analyzed by many authors.^{6,13,17} In particular, the vector of eigenvalues is found to be⁶:

$$\Lambda_p = \begin{bmatrix} v - \tilde{a}_p \\ v \\ v \\ v \\ v + \tilde{a}_p \end{bmatrix}, \quad (3.15)$$

where the waves corresponding to the eigenvalue v represent one mass fraction wave and one entropy wave for each phase. Herein, \tilde{a}_p as given by (3.9), is a well-known, classic expression sometimes referred to as the “Wood speed of sound”.¹⁷

3.2.1. The subcharacteristic condition

We now wish to prove that the pressure-relaxed model satisfies the subcharacteristic condition, in the sense of Definition 1, with respect to the basic model of Section 2. From (2.53) and (3.9) we obtain

$$\tilde{a}_p^{-2} = \tilde{a}_0^{-2} + Z_p^0, \quad (3.16)$$

where

$$Z_p^0 = \tilde{a}_0^{-2} \frac{\alpha_g \alpha_\ell}{\rho_g \rho_\ell c_g^2 c_\ell^2} (\rho_\ell c_\ell^2 - \rho_g c_g^2)^2. \quad (3.17)$$

Proposition 4. *The relaxed model (3.10)–(3.14) satisfies the subcharacteristic condition with respect to the basic relaxation model of Section 2, subject only to the physically fundamental conditions*

$$\rho_k > 0, \quad (3.18)$$

$$c_k^2 > 0, \quad (3.19)$$

for $k \in \{g, \ell\}$.

Proof. By (2.52) and (3.15), we observe that the interlacing condition of Definition 1 reduces to

$$\tilde{a}_0 \geq \tilde{a}_p, \quad (3.20)$$

which follows directly from (3.16)–(3.19) and the fact that

$$\alpha_k \in [0, 1] \quad k \in \{g, \ell\}. \quad (3.21)$$

□

4. Pressure and Temperature Relaxation

We now consider pT -relaxation, i.e. the simultaneous relaxation of both the pressure and temperature in the basic model of Section 2. This corresponds to taking the limit

$$\mathcal{H} \rightarrow \infty \quad (4.1)$$

in the model (3.10)–(3.14). We expect this limit to be equivalent to making the assumption

$$T_g = T_\ell = T \quad (4.2)$$

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and replacing (3.13)–(3.14) with their sum. We thus obtain the following model:

- *Mass conservation:*

$$\frac{\partial}{\partial t}(\rho_g \alpha_g) + \frac{\partial}{\partial x}(\rho_g \alpha_g v) = \mathcal{K}(\mu_\ell - \mu_g), \quad (4.3)$$

$$\frac{\partial}{\partial t}(\rho_\ell \alpha_\ell) + \frac{\partial}{\partial x}(\rho_\ell \alpha_\ell v) = \mathcal{K}(\mu_g - \mu_\ell). \quad (4.4)$$

- *Momentum conservation:*

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + p) = 0. \quad (4.5)$$

- *Energy conservation:*

$$\frac{\partial}{\partial t}(E_g + E_\ell) + \frac{\partial}{\partial x}(v(E_g + E_\ell + p)) = 0. \quad (4.6)$$

4.1. Wave velocities

The general N -component version of this model was analyzed in Ref. 6 for the non-stiff limit $\mathcal{K} \rightarrow 0$. The vector of eigenvalues corresponding to $N = 2$ was found to be

$$\Lambda_{pT} = \begin{bmatrix} v - \tilde{a}_{pT} \\ v \\ v \\ v + \tilde{a}_{pT} \end{bmatrix}. \quad (4.7)$$

Here the waves corresponding to the eigenvalue v represent one mass fraction wave and one mixture entropy wave. Furthermore, the mixture sound velocity was found as

$$\tilde{a}_{pT}^{-2} = \tilde{a}_p^{-2} + Z_{pT}^p, \quad (4.8)$$

where

$$Z_{pT}^p = \rho T \frac{C_{p,g} C_{p,\ell}}{C_{p,g} + C_{p,\ell}} \left(\frac{\Gamma_\ell}{\rho_\ell c_\ell^2} - \frac{\Gamma_g}{\rho_g c_g^2} \right)^2. \quad (4.9)$$

Herein, for $k \in \{g, \ell\}$, $C_{p,k}$ are the extensive heat capacities

$$C_{p,k} = \rho_k \alpha_k c_{p,k}, \quad (4.10)$$

where

$$c_{p,k} = T \left(\frac{\partial s_k}{\partial T} \right)_p. \quad (4.11)$$

Proposition 5. *In the sense of Definition 1, the pT -relaxed model (4.3)–(4.6) satisfies the subcharacteristic condition with respect to the p -relaxed model of Section 3,*

subject only to the physically fundamental constraints

$$\rho > 0, \quad (4.12)$$

$$T > 0, \quad (4.13)$$

$$c_{p,k} > 0. \quad (4.14)$$

The proof and further discussion may be found in Ref. 6.

5. Pressure and Material Relaxation

In this section, we investigate the effect of $p\mu$ -relaxation, i.e. simultaneous relaxation of pressure and chemical potential in the basic model presented in Section 2. This corresponds to taking the limit $\mathcal{K} \rightarrow \infty$ in the model (3.10)-(3.14). We expect this limit to correspond to the assumption that

$$\mu_g = \mu_\ell = \mu. \quad (5.1)$$

From (2.27) we obtain

$$\mu^* = \mu. \quad (5.2)$$

Before proceeding to the model equations, we calculate some useful differentials. Since the chemical potentials are equal, we know that $d\mu_g = d\mu_\ell = d\mu$. The chemical potential is given by $\mu_k = e_k + p/\rho_k - T_k s_k$, which, together with the fundamental thermodynamic differential (2.14), yields

$$d\mu = \frac{1}{\rho_\ell} dp - s_\ell dT_\ell = \frac{1}{\rho_g} dp - s_g dT_g. \quad (5.3)$$

We replace the temperature differential using

$$dT_k = \frac{T_k \Gamma_k}{\rho_k c_k^2} dp + \frac{T_k}{c_{p,k}} ds_k. \quad (5.4)$$

The pressure differential dp can be written as

$$dp = c_k^2 d\rho_k + \rho_k T_k \Gamma_k ds_k, \quad k \in \{g, \ell\} \quad (5.5)$$

Using the differentials (5.3)–(5.5), we can solve for $d\rho_k$,

$$d\rho_k = \frac{1}{c_k^2} (P_\ell ds_\ell + P_g ds_g - \rho_k T_k \Gamma_k ds_k), \quad (5.6)$$

where

$$P_g \equiv \left(\frac{\partial p}{\partial s_g} \right)_{s_\ell} = -\frac{s_g T_g \rho_g \rho_\ell}{c_{p,g}} \left(\rho_g - \rho_\ell + \rho_g \rho_\ell \left(\frac{s_g T_g \Gamma_g}{\rho_g c_g^2} - \frac{s_\ell T_\ell \Gamma_\ell}{\rho_\ell c_\ell^2} \right) \right)^{-1}, \quad (5.7)$$

$$P_\ell \equiv \left(\frac{\partial p}{\partial s_\ell} \right)_{s_g} = \frac{s_\ell T_\ell \rho_g \rho_\ell}{c_{p,\ell}} \left(\rho_g - \rho_\ell + \rho_g \rho_\ell \left(\frac{s_g T_g \Gamma_g}{\rho_g c_g^2} - \frac{s_\ell T_\ell \Gamma_\ell}{\rho_\ell c_\ell^2} \right) \right)^{-1}. \quad (5.8)$$

Solving the differential of the mixture density for $d\alpha_g$ yields

$$d\alpha_g = \frac{d\rho - \alpha_g d\rho_g - \alpha_\ell d\rho_\ell}{\rho_g - \rho_\ell} = -d\alpha_\ell, \quad (5.9)$$

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while the phase mass differential is

$$dm_k = \rho_k d\alpha_k + \alpha_k d\rho_k. \quad (5.10)$$

Now we insert (5.6) into (5.9) and (5.10), and abbreviate with \tilde{a}_p , so the mass differentials become

$$\begin{aligned} dm_g = \frac{\rho_g}{\rho_g - \rho_\ell} d\rho - \frac{\rho_g \rho_\ell}{\rho_g - \rho_\ell} \left(\frac{P_g \tilde{a}_p^{-2}}{\rho} - \alpha_g \frac{T_g \Gamma_g}{c_g^2} \right) ds_g \\ - \frac{\rho_g \rho_\ell}{\rho_g - \rho_\ell} \left(\frac{P_\ell \tilde{a}_p^{-2}}{\rho} - \alpha_\ell \frac{T_\ell \Gamma_\ell}{c_\ell^2} \right) ds_\ell, \end{aligned} \quad (5.11)$$

$$\begin{aligned} dm_\ell = \frac{\rho_\ell}{\rho_\ell - \rho_g} d\rho - \frac{\rho_g \rho_\ell}{\rho_\ell - \rho_g} \left(\frac{P_g \tilde{a}_p^{-2}}{\rho} - \alpha_g \frac{T_g \Gamma_g}{c_g^2} \right) ds_g \\ - \frac{\rho_g \rho_\ell}{\rho_\ell - \rho_g} \left(\frac{P_\ell \tilde{a}_p^{-2}}{\rho} - \alpha_\ell \frac{T_\ell \Gamma_\ell}{c_\ell^2} \right) ds_\ell. \end{aligned} \quad (5.12)$$

Finally, we have use for the velocity differential

$$dv = \frac{1}{\rho} (d(\rho v) - v d\rho). \quad (5.13)$$

5.1. Mass evolution equations

We now take the entropy equations (2.48)–(2.49) from the basic model and let $\mathcal{J}, \mathcal{K} \rightarrow \infty$, which corresponds to pressure and material relaxation, which yields

$$D_t s_g = \frac{\mathcal{H}}{\rho_g \alpha_g} \frac{T_\ell - T_g}{T_g} - \frac{s_g}{\rho_g \alpha_g} \left(\frac{\partial \alpha_g \rho_g}{\partial t} + \frac{\partial \alpha_g \rho_g v}{\partial x} \right), \quad (5.14)$$

$$D_t s_\ell = \frac{\mathcal{H}}{\rho_\ell \alpha_\ell} \frac{T_g - T_\ell}{T_\ell} - \frac{s_\ell}{\rho_\ell \alpha_\ell} \left(\frac{\partial \alpha_\ell \rho_\ell}{\partial t} + \frac{\partial \alpha_\ell \rho_\ell v}{\partial x} \right), \quad (5.15)$$

where we have replaced the chemical potential relaxation term using the mass conservation equations (2.1)–(2.2). By using the product rule for derivatives on $\partial_x(\alpha_k \rho_k v)$ and the differentials (5.11)–(5.13), we can reformulate the mass equations as

$$\begin{aligned} \frac{\partial m_g}{\partial t} + \frac{\partial}{\partial x}(m_g v) = \frac{\partial v}{\partial x} \frac{\rho \tilde{a}_{p\mu}^2}{P_g} \frac{\rho_g \alpha_g s_g T_g C_{p,\ell}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \\ - \mathcal{H}(T_g - T_\ell) \left(\frac{\tilde{a}_{p\mu}}{\tilde{a}_p} \right)^2 \frac{s_g C_{p,\ell} + s_\ell C_{p,g}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \\ + \mathcal{H}(T_g - T_\ell) \left(\frac{\Gamma_g}{\rho_g c_g^2} - \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} \right) \cdot \frac{\rho \tilde{a}_{p\mu}^2}{P_g} \frac{\rho_g \alpha_g s_g T_g C_{p,\ell}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}}, \end{aligned} \quad (5.16)$$

$$\begin{aligned}
 \frac{\partial m_\ell}{\partial t} + \frac{\partial}{\partial x}(m_\ell v) &= \frac{\partial v}{\partial x} \frac{\rho \tilde{a}_{p\mu}^2}{P_\ell} \frac{\rho_\ell \alpha_\ell s_\ell T_\ell C_{p,g}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \\
 &\quad + \mathcal{H}(T_g - T_\ell) \left(\frac{\tilde{a}_{p\mu}}{\tilde{a}_p} \right)^2 \frac{s_g C_{p,\ell} + s_\ell C_{p,g}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \\
 &\quad + \mathcal{H}(T_g - T_\ell) \left(\frac{\Gamma_g}{\rho_g c_g^2} - \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} \right) \cdot \frac{\rho \tilde{a}_{p\mu}^2}{P_\ell} \frac{\rho_\ell \alpha_\ell s_\ell T_\ell C_{p,g}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}}, \quad (5.17)
 \end{aligned}$$

where

$$\tilde{a}_{p\mu}^{-2} = \tilde{a}_p^{-2} + \frac{\rho C_{p,g} C_{p,\ell}}{\rho_g^2 \rho_\ell^2 (C_{p,\ell} s_g^2 T_g + C_{p,g} s_\ell^2 T_\ell)} \left(\rho_g - \rho_\ell + \rho_g \rho_\ell \left(s_g \frac{T_g \Gamma_g}{\rho_g c_g^2} - s_\ell \frac{T_\ell \Gamma_\ell}{\rho_\ell c_\ell^2} \right) \right)^2. \quad (5.18)$$

5.2. Entropy evolution equations

Substituting (5.16)–(5.17) into (5.14)–(5.15), we obtain

$$\begin{aligned}
 D_t s_g + \frac{\rho s_g^2 T_g C_{p,\ell}}{P_g (s_\ell^2 T_\ell C_{p,g} + s_g^2 T_g C_{p,\ell})} \tilde{a}_{p\mu}^2 \frac{\partial v}{\partial x} \\
 &= \frac{\mathcal{H}}{\rho_g \alpha_g} \frac{T_\ell - T_g}{T_g} \left(1 - s_g T_g \left(\frac{\tilde{a}_{p\mu}}{\tilde{a}_p} \right)^2 \frac{s_g C_{p,\ell} + s_\ell C_{p,g}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \right) \\
 &\quad + \mathcal{H}(T_\ell - T_g) \left(\frac{\Gamma_g}{\rho_g c_g^2} - \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} \right) \cdot \frac{\rho \tilde{a}_{p\mu}^2}{P_g} \frac{s_g^2 T_g C_{p,\ell}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \quad (5.19)
 \end{aligned}$$

and

$$\begin{aligned}
 D_t s_\ell + \frac{\rho s_\ell^2 T_\ell C_{p,g}}{P_\ell (s_\ell^2 T_\ell C_{p,g} + s_g^2 T_g C_{p,\ell})} \tilde{a}_{p\mu}^2 \frac{\partial v}{\partial x} \\
 &= \frac{\mathcal{H}}{\rho_\ell \alpha_\ell} \frac{T_g - T_\ell}{T_\ell} \left(1 - s_\ell T_\ell \left(\frac{\tilde{a}_{p\mu}}{\tilde{a}_p} \right)^2 \frac{s_g C_{p,\ell} + s_\ell C_{p,g}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \right) \\
 &\quad - \mathcal{H}(T_g - T_\ell) \left(\frac{\Gamma_g}{\rho_g c_g^2} - \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} \right) \cdot \frac{\rho \tilde{a}_{p\mu}^2}{P_\ell} \frac{s_\ell^2 T_\ell C_{p,g}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}}. \quad (5.20)
 \end{aligned}$$

5.3. Energy evolution equations

Using (5.1)–(5.2), we may write the energy equations (3.13)–(3.14) as

$$\begin{aligned}
 & \frac{\partial E_g}{\partial t} + \frac{\partial}{\partial x} (vE_g) + \frac{\rho_g \alpha_g}{\rho} v \frac{\partial p}{\partial x} + p \alpha_g \frac{\rho \tilde{a}_p^2}{\rho_g c_g^2} \frac{\partial v}{\partial x} \\
 & - \frac{\rho \tilde{a}_{p\mu}^2}{P_g} \frac{\rho_g \alpha_g s_g T_g C_{p,\ell}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \left(\mu + \frac{1}{2} v^2 + p \frac{\alpha_g (\Gamma_\ell s_\ell T_\ell - c_\ell^2) + \alpha_\ell (\Gamma_g s_g T_g - c_g^2)}{\rho_g \alpha_\ell c_g^2 + \rho_\ell \alpha_g c_\ell^2} \right) \frac{\partial v}{\partial x} \\
 & = \left(1 - p \frac{\alpha_g \Gamma_\ell + \alpha_\ell \Gamma_g}{\rho_g \alpha_\ell c_g^2 + \rho_\ell \alpha_g c_\ell^2} \right) \mathcal{H}(T_\ell - T_g) \\
 & + \mathcal{H}(T_\ell - T_g) \left(\mu + \frac{1}{2} v^2 + p \frac{\alpha_g (\Gamma_\ell s_\ell T_\ell - c_\ell^2) + \alpha_\ell (\Gamma_g s_g T_g - c_g^2)}{\rho_g \alpha_\ell c_g^2 + \rho_\ell \alpha_g c_\ell^2} \right) \\
 & \cdot \left(\frac{s_g C_{p,\ell} + s_\ell C_{p,g}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \left(\frac{\tilde{a}_{p\mu}}{\tilde{a}_p} \right)^2 - \left(\frac{\Gamma_g}{\rho_g c_g^2} - \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} \right) \cdot \frac{\rho \tilde{a}_{p\mu}^2}{P_g} \frac{\rho_g \alpha_g s_g T_g C_{p,\ell}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \right), \tag{5.21}
 \end{aligned}$$

$$\begin{aligned}
 & \frac{\partial E_\ell}{\partial t} + \frac{\partial}{\partial x} (vE_\ell) + \frac{\rho_\ell \alpha_\ell}{\rho} v \frac{\partial p}{\partial x} + p \alpha_\ell \frac{\rho \tilde{a}_p^2}{\rho_\ell c_\ell^2} \frac{\partial v}{\partial x} \\
 & - \frac{\rho \tilde{a}_{p\mu}^2}{P_\ell} \frac{\rho_\ell \alpha_\ell s_\ell T_\ell C_{p,g}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \left(\mu + \frac{1}{2} v^2 + p \frac{\alpha_g (\Gamma_\ell s_\ell T_\ell - c_\ell^2) + \alpha_\ell (\Gamma_g s_g T_g - c_g^2)}{\rho_g \alpha_\ell c_g^2 + \rho_\ell \alpha_g c_\ell^2} \right) \frac{\partial v}{\partial x} \\
 & = \left(1 - p \frac{\alpha_g \Gamma_\ell + \alpha_\ell \Gamma_g}{\rho_g \alpha_\ell c_g^2 + \rho_\ell \alpha_g c_\ell^2} \right) \mathcal{H}(T_g - T_\ell) \\
 & + \mathcal{H}(T_g - T_\ell) \left(\mu + \frac{1}{2} v^2 + p \frac{\alpha_g (\Gamma_\ell s_\ell T_\ell - c_\ell^2) + \alpha_\ell (\Gamma_g s_g T_g - c_g^2)}{\rho_g \alpha_\ell c_g^2 + \rho_\ell \alpha_g c_\ell^2} \right) \\
 & \cdot \left(\frac{s_g C_{p,\ell} + s_\ell C_{p,g}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \left(\frac{\tilde{a}_{p\mu}}{\tilde{a}_p} \right)^2 + \left(\frac{\Gamma_g}{\rho_g c_g^2} - \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} \right) \cdot \frac{\rho \tilde{a}_{p\mu}^2}{P_\ell} \frac{\rho_\ell \alpha_\ell s_\ell T_\ell C_{p,g}}{s_g^2 T_g C_{p,\ell} + s_\ell^2 T_\ell C_{p,g}} \right), \tag{5.22}
 \end{aligned}$$

where we have replaced the material relaxation term using the mass conservation equations (5.16)–(5.17).

5.4. The $p\mu$ -relaxed model

The relaxed model with respect to phase and volume transfer may now be written out in a simple form as follows:

- *Mass conservation:*

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0. \tag{5.23}$$

- *Momentum conservation:*

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + p) = 0. \quad (5.24)$$

- *Entropy evolution:*

$$T_g \left(\frac{\partial}{\partial t}(\rho_g \alpha_g s_g) + \frac{\partial}{\partial x}(\rho_g \alpha_g s_g v) \right) = \mathcal{H}(T_\ell - T_g), \quad (5.25)$$

$$T_\ell \left(\frac{\partial}{\partial t}(\rho_\ell \alpha_\ell s_\ell) + \frac{\partial}{\partial x}(\rho_\ell \alpha_\ell s_\ell v) \right) = \mathcal{H}(T_g - T_\ell). \quad (5.26)$$

Here we have used the relaxed versions of (2.18)–(2.19). Alternatively, one could use the equivalent, but more obscure formulation (5.21)–(5.22).

5.5. Wave velocities

In this section, we wish to derive the wave velocities for the model (5.23)–(5.26) in the non-stiff limit, i.e. when $\mathcal{H} \rightarrow 0$.

Proposition 6. *The vector of eigenvalues of the pressure-material relaxation model (5.23)–(5.26) is given by*

$$\mathbf{\Lambda}_{p\mu} = \begin{bmatrix} v - \tilde{a}_{p\mu} \\ v \\ v \\ v + \tilde{a}_{p\mu} \end{bmatrix}, \quad (5.27)$$

where

$$\tilde{a}_{p\mu}^{-2} = \tilde{a}_p^{-2} + Z_{p\mu}^p \quad (5.28)$$

is the mixture sound velocity, as defined in (5.18), where

$$Z_{p\mu}^p = \frac{\rho C_{p,g} C_{p,\ell}}{\rho_g^2 \rho_\ell^2 (C_{p,\ell} s_g^2 T_g + C_{p,g} s_\ell^2 T_\ell)} \left(\rho_g - \rho_\ell + \rho_g \rho_\ell \left(s_g \frac{T_g \Gamma_g}{\rho_g c_g^2} - s_\ell \frac{T_\ell \Gamma_\ell}{\rho_\ell c_\ell^2} \right) \right)^2. \quad (5.29)$$

Proof. By using the differentials (5.3)–(5.4) and (5.13), we can rewrite the system in a quasi-linear form,

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho v \\ s_g \\ s_\ell \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ -v^2 & 2v & P_g & P_\ell \\ -V_g v & V_g & v & 0 \\ -V_\ell v & V_\ell & 0 & v \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \rho \\ \rho v \\ s_g \\ s_\ell \end{bmatrix} = \mathbf{S}, \quad (5.30)$$

where

$$V_g = \frac{s_g^2 T_g C_{p,\ell}}{P_g (s_\ell^2 T_\ell C_{p,g} + s_g^2 T_g C_{p,\ell})} \tilde{a}_{p\mu}^2, \quad (5.31)$$

$$V_\ell = \frac{s_\ell^2 T_\ell C_{p,g}}{P_\ell (s_\ell^2 T_\ell C_{p,g} + s_g^2 T_g C_{p,\ell})} \tilde{a}_{p\mu}^2. \quad (5.32)$$

Solving for the eigenvalues of this system yields Eq. (5.18). \square

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5.5.1. *The subcharacteristic condition*

We now wish to show that the subcharacteristic condition is satisfied with respect to the pressure relaxation model.

Proposition 7. *The pressure-material relaxed model (5.23)–(5.26) satisfies the subcharacteristic condition with respect to the pressure relaxation model of Section 3, subject only to the physically fundamental conditions*

$$\rho_k > 0, \tag{5.33}$$

$$c_k^2 > 0, \tag{5.34}$$

$$c_{p,k} > 0, \tag{5.35}$$

$$T_k > 0, \tag{5.36}$$

for $k \in \{g, \ell\}$.

Proof. By (3.15) and (5.27), we observe that the interlacing condition of Definition 1 reduces to

$$\tilde{a}_p \geq \tilde{a}_{p\mu}, \tag{5.37}$$

which follows directly from (5.28)–(5.29) and the fact that the volume fractions α_k are within the unit interval. \square

6. Full Relaxation

In this section, we consider the model that arises from assuming simultaneous equilibrium in both volume, heat and phase transfer. This *homogeneous equilibrium* model is expected to correspond to the limit $\mathcal{K} \rightarrow \infty$ in the model of Section 4 and the limit $\mathcal{H} \rightarrow \infty$ in the model of Section 5.

We make the full equilibrium assumptions

$$p_g = p_\ell, \tag{6.1}$$

$$T_g = T_\ell, \tag{6.2}$$

$$\mu_g = \mu_\ell, \tag{6.3}$$

and formulate conservation equations for total mass, momentum and energy:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{6.4}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + p) = 0, \tag{6.5}$$

$$\frac{\partial}{\partial t}(E_g + E_\ell) + \frac{\partial}{\partial x}(v(E_g + E_\ell + p)) = 0. \tag{6.6}$$

6.1. Wave velocities

The homogeneous equilibrium model is identical in structure to the Euler equations for gas dynamics; consequently, the model has a wave structure given by

$$\Lambda_{pT\mu} = \begin{bmatrix} v - \tilde{a}_{pT\mu} \\ v \\ v + \tilde{a}_{pT\mu} \end{bmatrix}, \quad (6.7)$$

where the wave corresponding to the eigenvalue v transports the mixture entropy. Saurel et al.¹⁷ state the following expression for $\tilde{a}_{pT\mu}$:

$$\tilde{a}_{pT\mu}^{-2} = \tilde{a}_p^{-2} + \rho T \left[\frac{\rho_g \alpha_g}{c_{p,g}} \left(\frac{\partial s_g}{\partial p} \right)_{\text{sat}}^2 + \frac{\rho_\ell \alpha_\ell}{c_{p,\ell}} \left(\frac{\partial s_\ell}{\partial p} \right)_{\text{sat}}^2 \right]. \quad (6.8)$$

Herein, the symbol $(\cdot)_{\text{sat}}$ denotes differentiation along the boiling point curve, where

$$\mu_g = \mu_\ell. \quad (6.9)$$

6.2. The subcharacteristic condition with respect to pT -relaxation

In this section, we aim to show that the full-relaxation model satisfies the subcharacteristic condition with respect to the pressure-temperature-relaxation model of Section 4. To this end, we rewrite the entropy derivatives,

$$\left(\frac{\partial s_k}{\partial p} \right)_{\text{sat}} = \left(\frac{\partial s_k}{\partial p} \right)_T + \left(\frac{\partial s_k}{\partial T} \right)_p \left(\frac{\partial T}{\partial p} \right)_{\text{sat}}. \quad (6.10)$$

By using the fundamental thermodynamic differential (2.14), the definition of the sound velocity (2.54), the triple product rule and the Maxwell relation $(\partial \rho / \partial s)_p = -\rho^2 (\partial T / \partial p)_s$, we find that

$$\left(\frac{\partial T}{\partial p} \right)_{s_k} = \frac{\Gamma_k T}{\rho_k c_k^2}. \quad (6.11)$$

The temperature derivative at the boiling curve is rewritten using the Clausius-Clapeyron relation,

$$\left(\frac{\partial T}{\partial p} \right)_{\text{sat}} = -\frac{T(\rho_g - \rho_\ell)}{(h_g - h_\ell)\rho_g \rho_\ell}. \quad (6.12)$$

We use the triple product rule once more, together with (4.11) and (6.11)–(6.12), which yields

$$\left(\frac{\partial s_k}{\partial p} \right)_{\text{sat}} = -\frac{\Gamma_k c_{p,k}}{\rho_k c_k^2} - \frac{c_{p,k}(\rho_g - \rho_\ell)}{\rho_g \rho_\ell (h_g - h_\ell)}. \quad (6.13)$$

Inserting Eq. (6.13) into the sound velocity expression (6.8) and rearranging terms yields

$$\tilde{a}_{pT\mu}^{-2} = \tilde{a}_{pT}^{-2} + Z_{pT\mu}^{pT}, \quad (6.14)$$

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where

$$Z_{pT\mu}^{pT} = \frac{\rho T}{C_{p,g} + C_{p,\ell}} \left(\frac{\rho_g - \rho_\ell}{\rho_g \rho_\ell (h_g - h_\ell)} (C_{p,g} + C_{p,\ell}) + \frac{\Gamma_g C_{p,g}}{\rho_g c_g^2} + \frac{\Gamma_\ell C_{p,\ell}}{\rho_\ell c_\ell^2} \right)^2. \quad (6.15)$$

Proposition 8. *The homogeneous equilibrium model (6.4)–(6.6) satisfies the sub-characteristic condition with respect to the pressure-temperature relaxation model of Section 4, subject only to the physically fundamental conditions*

$$\rho_k > 0, \quad (6.16)$$

$$c_{p,k} > 0, \quad (6.17)$$

$$T > 0, \quad (6.18)$$

for $k \in \{g, \ell\}$.

Proof. By (4.7) and (6.7), we observe that the interlacing condition of Definition 1 reduces to

$$\tilde{a}_{pT} \geq \tilde{a}_{pT\mu}, \quad (6.19)$$

which follows directly from (6.14)–(6.15) and the fact that the volume fractions α_k are within the unit interval. \square

6.3. The subcharacteristic condition with respect to $p\mu$ -relaxation

In this section, we aim to show that the full-relaxation model satisfies the sub-characteristic condition with respect to the pressure-material relaxation model of Section 5. We take (6.14) as our starting point, and by simply rearranging the terms in a new manner, we get

$$\tilde{a}_{pT\mu}^{-2} = \tilde{a}_{p\mu}^{-2} + Z_{pT\mu}^{p\mu}, \quad (6.20)$$

where

$$Z_{pT\mu}^{p\mu} = \frac{\rho}{T(C_{p,\ell} s_g^2 + C_{p,g} s_\ell^2)} \left(\frac{(\rho_\ell - \rho_g)(C_{p,g} s_\ell + C_{p,\ell} s_g)}{\rho_g \rho_\ell (s_\ell - s_g)} + T \frac{C_{p,g} C_{p,\ell} s_g s_\ell \left(\frac{\Gamma_\ell}{\rho_\ell c_\ell^2} + \frac{\Gamma_g}{\rho_g c_g^2} \right) + \frac{\Gamma_g}{\rho_g c_g^2} C_{p,g}^2 s_\ell^2 + \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} C_{p,\ell}^2 s_g^2}{C_{p,g} s_\ell + C_{p,\ell} s_g} \right)^2. \quad (6.21)$$

Proposition 9. *The homogeneous equilibrium model (6.4)–(6.6) satisfies the sub-characteristic condition with respect to the pressure-temperature relaxation model of Section 5, subject only to the physically fundamental conditions*

$$\rho_k > 0, \quad (6.22)$$

$$c_{p,k} > 0, \quad (6.23)$$

$$T > 0, \quad (6.24)$$

for $k \in \{g, \ell\}$.

Proof. By (5.27) and (6.7), we observe that the interlacing condition of Definition 1 reduces to

$$\tilde{a}_{p\mu} \geq \tilde{a}_{pT\mu}, \quad (6.25)$$

which follows directly from (6.20)–(6.21) and the fact that the volume fractions α_k are within the unit interval. \square

Remark 1.

We have now considered 5 different relaxation processes, connecting 5 different models as described in Figure 1. From (2.53) we see that the sound velocity of the basic model reduces to the standard one-phase sound velocities in the limit where one phase disappears; more precisely,

$$\lim_{\alpha_k \rightarrow 1} \tilde{a}_0 = c_k \quad k \in \{g, \ell\}, \quad (6.26)$$

where c_k is given by (2.54). Furthermore, we obtain from (3.17), (4.9) and (5.29):

$$\lim_{\alpha_k \rightarrow 1} Z_p^0 = \lim_{\alpha_k \rightarrow 1} Z_{pT}^p = \lim_{\alpha_k \rightarrow 1} Z_{p\mu}^p = 0, \quad k \in \{g, \ell\}. \quad (6.27)$$

Consequently, p -relaxation, pT -relaxation and $p\mu$ -relaxation all yield continuous sound velocities in the one-phase limit.

However, from (6.15) and (6.21) we obtain

$$\lim_{\alpha_g \rightarrow 1} Z_{pT\mu}^{pT} = \lim_{\alpha_g \rightarrow 1} Z_{pT\mu}^{p\mu} = c_{p,g} T \left(\frac{\rho_g - \rho_\ell}{\rho_\ell (h_g - h_\ell)} + \frac{\Gamma_g}{c_g^2} \right)^2, \quad (6.28)$$

$$\lim_{\alpha_\ell \rightarrow 1} Z_{pT\mu}^{pT} = \lim_{\alpha_\ell \rightarrow 1} Z_{pT\mu}^{p\mu} = c_{p,\ell} T \left(\frac{\rho_g - \rho_\ell}{\rho_g (h_g - h_\ell)} + \frac{\Gamma_\ell}{c_\ell^2} \right)^2. \quad (6.29)$$

Hence the sound velocity is **discontinuous** in the one-phase limit when thermal and material relaxation are applied simultaneously, despite the fact that none of these processes alone give rise to discontinuous behaviour. This result will be illustrated in the following.

7. Illustrative Examples

In this section, we illustrate our results for some two-phase mixtures of practical relevance.

7.1. Water and steam

We first consider a mixture of water and steam under atmospheric pressure; relevant thermodynamic parameters representing this case are stated in Table 1. The mixture sound velocities as a function of gas volume fraction are plotted in Figure 2. In Figure 2(a), the sound velocities given by (2.53), (3.9), (4.8), (5.18) and (6.8) are

Table 1. Water-steam mixture under atmospheric pressure.

Quantity	gas (steam)	liquid (water)
Pressure (MPa)	0.1	0.1
Temperature (K)	372.76	372.76
Density (kg/m ³)	0.59031	958.64
Sound speed (m/s)	472.05	1543.4
c_p (J/(kg · K))	2075.9	4216.1
Entropy (m ² /(s ² · K))	7358.8	1302.6
Γ (dimensionless)	0.33699	0.4

Table 2. Two-phase CO₂ mixture.

Quantity	gas (CO ₂)	liquid (CO ₂)
Pressure (MPa)	5	5
Temperature (K)	287.43	287.43
Density (kg/m ³)	156.71	827.21
Sound speed (m/s)	201.54	398.89
c_p (J/(kg · K))	3138.0	3356.9
Entropy (m ² /(s ² · K))	1753.9	1128.8
Γ (dimensionless)	0.30949	0.63175

plotted for the full velocity range. We note that p -relaxation has a very strong effect on the sound velocity, which is reduced by more than an order of magnitude in the two-phase region. The effect of the other relaxation procedures are hard to discern from Figure 2(a).

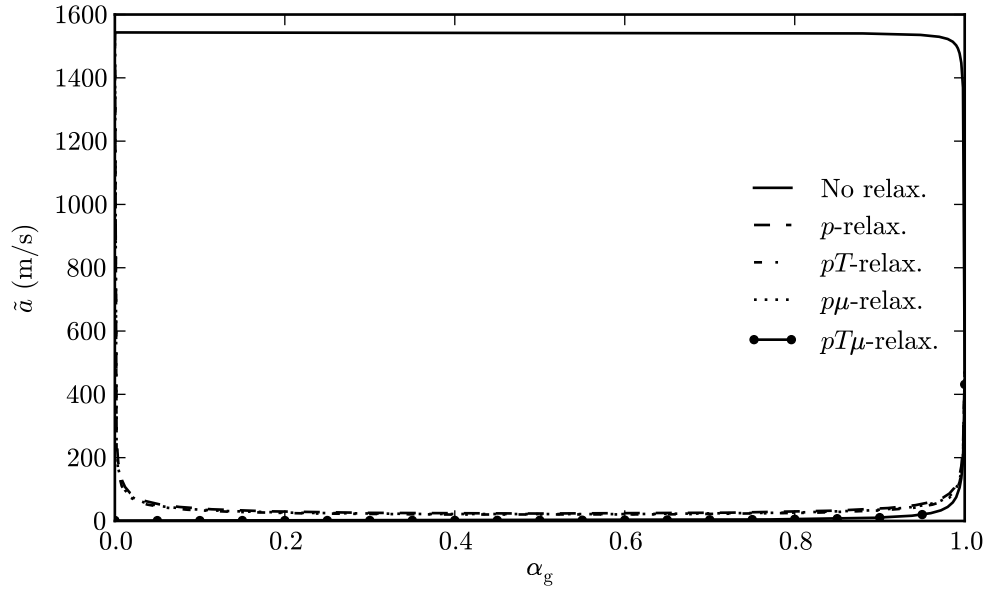
In Figure 2(b), we focus on the physically interesting part of the plot. We observe that μ -relaxation and T -relaxation has little effect on the mixture sound velocity when applied separately to the p -relaxed model.

However, when applied *simultaneously*, their effect is significant. The mixture sound velocity is strongly reduced in the two-phase region. We also observe the discontinuity in the one-phase limits, as stated by (6.28)–(6.29).

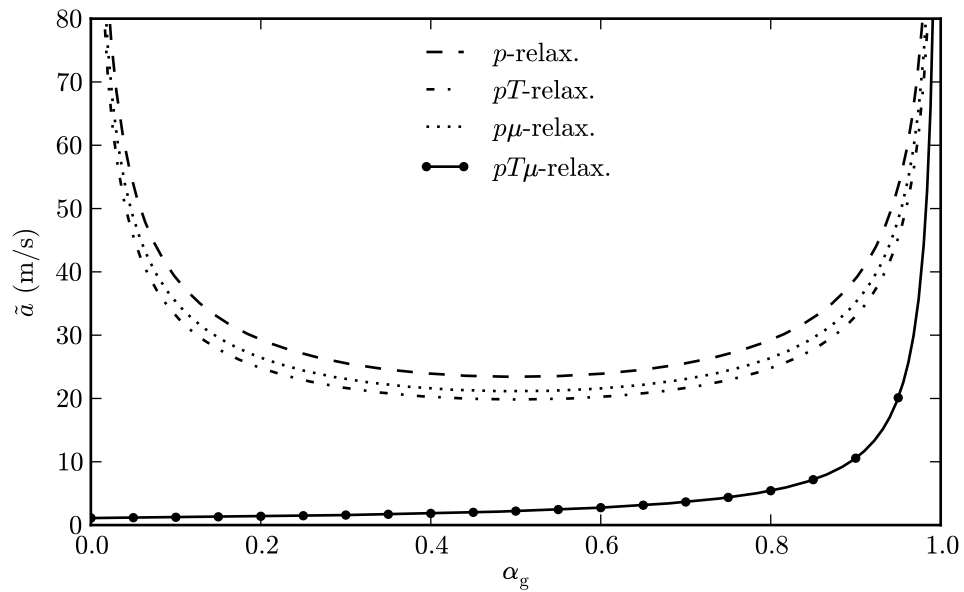
7.2. A two-phase CO₂ mixture

We here consider a CO₂ mixture under high pressure, representative of conditions applying to pipeline transport. Thermodynamic parameters modelling such a case are stated in Table 2.

The mixture sound velocities as a function of gas volume fraction are plotted in Figure 3. Here all relaxation procedures contribute to significantly reducing the mixture sound velocity. Note that just as for the water-steam mixture, the effects of pT -relaxation and $p\mu$ -relaxation are highly similar. For the full equilibrium velocity $\tilde{a}_{pT\mu}$, we observe the expected discontinuity in the one-phase limits. This is particularly pronounced for the transition to pure liquid.



(a) Full velocity range



(b) Magnification of the subrange $0 < \tilde{a} < 80$ m/s

Fig. 2. Mixture sound velocities for models with different relaxation conditions for water and steam at 1 bar and 100°C.

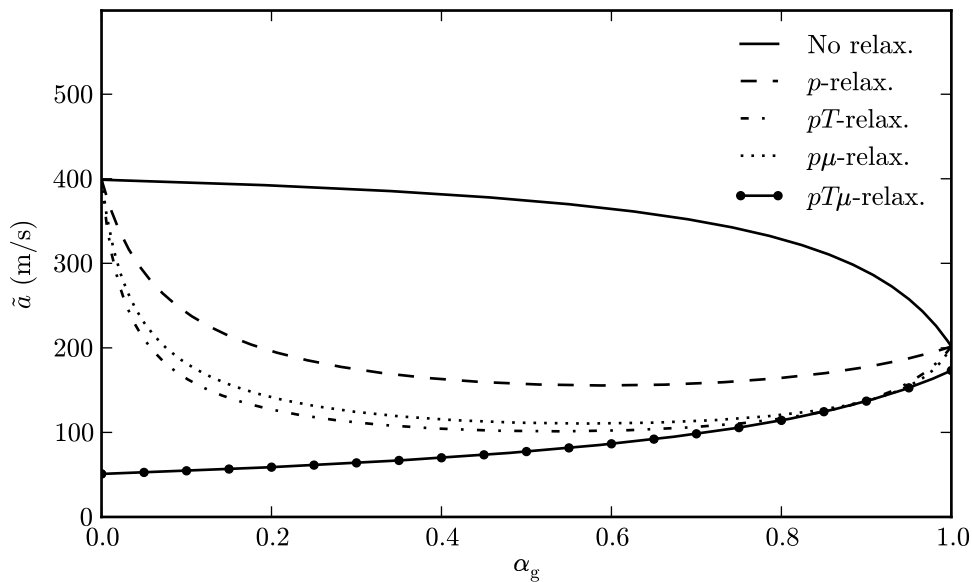


Fig. 3. Mixture sound velocities for models with different equilibrium conditions for carbon dioxide (CO_2) at the boiling point ($T=14.28^\circ\text{C}$) at 50 bar.

8. Summary

We have studied a hierarchy of relaxation model for two-phase flows, with a special emphasis on the interplay between the effects of heat and phase transfer. We have formulated explicit relaxation procedures for phase, heat and volume transfer that respect the first and second law of thermodynamics. We have attempted to make this formulation as general as possible.

By combining original and established results, we have systematically presented the wave structures of the models arising from different levels of relaxation. Our focus has been on the subcharacteristic condition. In this respect, we have provided a somewhat original, and perhaps heuristically useful formulation; the subcharacteristic condition manifests itself as a hierarchy of sums-of-squares modifications to the basic mixture sound velocity.

The fully relaxed equilibrium model gives rise to a discontinuous sound velocity in the limit where one phase disappears. A final interesting observation is that this phenomenon is not associated to any particular relaxation procedure, but arises in our formulation only from the *combination* of heat and phase transfer equilibrium.

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